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A Review of Gradient Algorithms for Numerical Computation of Optimal Trajectories

Wander Almodovar Golfetto[1,*], Sandro da Silva Fernandes[2]
[1]Departamento de Ciência e Tecnologia Aeroespacial – São José dos Campos/SP – Brazil
[2]Instituto Tecnológico de Aeronáutica – São José dos Campos/SP – Brazil

Abstract: *In this paper, two classic direct methods for numerical computation of optimal trajectories were revisited: the steepest descent method and the direct one based upon the second variation theory. The steepest descent method was developed for a Mayer problem of optimal control, with free final state and fixed terminal times. Terminal constraints on the state variables were considered through the penalty function method. The second method was based upon the theory of second variation and it involves the closed-loop solutions of a linear quadratic optimal control problem. The algorithm was developed for a Bolza problem of optimal control, with fixed terminal times and constrained initial and final states. Problems with free final time are also considered by using a transformation approach. An algorithm that combines the main characteristics of these methods was also presented. The methods were applied for solving two classic optimization problems – Brachistochrone and Zermelo – and their main advantages and disadvantages were discussed. Finally, the optimal space trajectories transference between coplanar circular orbits for different times of flight was calculated, using the proposed algorithm.*

Keywords: *Optimization of trajectories, Numerical methods, Steepest descent method, Second-order gradient method.*

INTRODUCTION

In the last four decades, a great number of numerical methods have been developed for computing solutions to optimal control problems. These methods are divided into two main classes: indirect and direct. The indirect ones involve the solution of a nonlinear two-point boundary value problem obtained from a set of necessary conditions for a local extremum of the objective function, provided by the Pontryagin's Maximum Principle (Pontryagin *et al.*, 1962) or the calculus of variations (Bliss, 1946; Hestenes, 1966). For instance, the quasilinearization method, also known as generalized Newton-Raphson method (McGill and Kenneth, 1964), and the shooting one (Sage and White, 1977) belong to this class of numerical methods. On the other hand, the direct methods use only equations of motion and terminal conditions and they attempt to minimize the objective function through an iterative descent procedure. The direct approach was first proposed by Kelley (1960) and is referred to as the gradient method.

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*author for correspondence: wanderag@gmail.com
Avenida Brigadeiro Faria Lima, 1.941 – Parque Martim Cererê
CEP 12.227-000 São José dos Campos/SP – Brazil

Other direct methods are the steepest descent method (Bryson and Denham, 1962) and the ones based upon the second variation theory (Bullock and Franklin, 1967; Longmuir and Bohn, 1969; Bryson and Ho, 1975). These direct methods use the calculus of variations, but there is another class of direct methods that transform optimal control problems into nonlinear programming ones, such as the direct transcription or collocation method (Betts, 1993; 1994) and the direct methods based on differential inclusion concepts (Seywald, 1994; Coverstone-Carroll and Williams, 1994).

In this paper, the steepest descent method and the direct method based upon the second variation theory, which will be referred to as second variation method, are discussed, considering different algorithms in comparison to the original ones proposed by Bryson and Denham (1962) and Longmuir and Bohn (1969). The steepest descent method was developed for a Mayer problem of optimal control, with free final state and fixed terminal times. Terminal constraints on state variables were considered through the penalty function method (Hestenes, 1969; O'Doherty and Pierson, 1974). A procedure to adjust the step size in control space is proposed to improve the convergence rate and avoid the divergence of the method

as the optimal solution is approached. On the other hand, the second order gradient method involves the determination of closed-loop solutions of a linear quadratic optimal control problem using the Riccati transformation. The algorithm was developed for a Bolza problem of optimal control, with fixed terminal times and constrained initial and final states. A slight modification of the algorithm was made to satisfy the Legendre condition and avoid the divergence of the method (Bullock and Franklin, 1967). In both methods, problems with free final time are treated by using a transformation approach that introduces a new independent variable and an additional control one, and transforms a time-free problem in a new time-fixed one (Hussu, 1972; Quintana and Davison, 1973). The methods are applied in solving two classic optimization problems – Brachistochrone and Zermelo problems –, and their main advantages and disadvantages are discussed. Finally, an algorithm that combines the main positive characteristics of these methods was presented and applied in space trajectories optimization for transference between coplanar circular orbits.

STEEPEST DESCENT METHOD

The steepest descent method is an iterative direct method widely used for computing a m-vector of control variables $u(t),\ t_0 \le t \le t_f$, which minimizes a scalar performance index in an optimization problem (Kelley, 1960; Bryson and Denham, 1962; McIntyre, 1968; McDermott and Fowler, 1977).

A simplified version of the steepest descent method was here presented for a Mayer problem of optimal control with free final state and fixed terminal times. Problems with constraints on the state variables at the final fixed time were treated by using the penalty function method (Hestenes, 1969; O'Doherty and Pierson, 1974), and those with free final time were treated by using a transformation approach, which introduces new independent and additional control variables (Hussu, 1972; Quintana and Davison, 1973). Details of this technique are further described. Thus, this simplified version has two main features: the algorithm is very simple, requiring a single numerical integration of the adjoint equations at each step, and some of the typical divergence problems described by McDermott and Fowler (1977) are circumvented.

Consider the system of differential equations (Eq. 1):

$$\frac{dx_i}{dt} = f_i(x,u), \quad i = 1,...,n, \tag{1}$$

where:

x is a n-vector of state variables; u is a m-vector of control variables; $f(.): R^n \times R^m \to R^n$, f_i (.) and $\partial f_i / \partial x_j$, i and $j = 1,...,n$ are defined and continuous on $R^n \times R^m$. It is assumed that there are no constraints on the state or control variables.

The problem consists in determining the control $u^*(t)$ that transfers the system (Eq. 1) from the initial conditions (Eq. 2):

$$x(t_0) = x_0, \tag{2}$$

to the final conditions at t_f (Eq. 3):

$$x(t_f) = \text{free}, \tag{3}$$

and minimizes the performance index (Eq. 4):

$$J[u] = g(x(t_f)), \tag{4}$$

with $g: R^n \to R$ and $\partial g / \partial x_i$, $i = 1,...,n$, continuous on R^n.

For completeness, a brief description of the steepest descent method is presented. The development of the algorithm was based on the classic calculus of variations (Bliss, 1946; Hestenes, 1966; Gelfand and Fomin, 1963; Elsgolts, 1977). Introducing the n-vector λ of Lagrangian multipliers – adjoint variables –, the augmented performance index is formed as Eq. 5:

$$J = g(x(t_f)) + \int_{t_0}^{t_f} \left[-H + \lambda^T \dot{x} \right] dt, \tag{5}$$

where, H is the Hamiltonian function (Eq. 6),

$$H(x,\lambda,u) = \lambda^T f(x,u). \tag{6}$$

Let $u^0(t),\ t_0 \le t \le t_f$ be an arbitrary starting approximation of the control $u^*(t),\ t_0 \le t \le t_f$ and $x^0(t),\ t_0 \le t \le t_f$ the corresponding trajectory, obtained by integrating the system (Eq. 1). Let $u^1(t) = u^0(t) + \delta u(t), t_0 \le t \le t_f$ be the second iterate such that $J[u^1] \le J[u^0]$. It is assumed that the control variation $\delta u(t),\ t_0 \le t \le t_f$ is small, in order that the linearity assumptions are valid, and they satisfy the constraint as in Eq. 7:

$$\int_{t_0}^{t_f} \delta u^T(\tau) W(\tau) \delta u(\tau) d\tau = K^2, \tag{7}$$

where, $W(\tau)$ is an arbitrary, time-varying, positive-definite symmetric $m \times m$ matrix of weighting functions chosen to improve convergence of the steepest descent method, and K is

the step size in control space. Both $W(\tau)$ and K must be chosen by the user of the algorithm.

The control variation $\delta u(t), t_0 \le t \le t_f$ causes perturbations in the state vector so that the corresponding trajectory to the control $u^1(t), t_0 \le t \le t_f$, can be expressed as $x^1(t) = x^0(t) + \delta x(t)$. Accordingly, the change in $\bar{J}$, which is assumed to be differentiable, to first order in the perturbations, is given by Eq. 8 (McIntyre, 1968):

$$\delta\bar{J} = \left[g_{x_f} + \lambda^T(t_f)\right]\delta x(t_f) + \int_{t_0}^{t_f} \left[\left[-H_x - \dot{\lambda}^T\right]\delta x - H_u \delta u\right], \quad (8)$$

since $\delta x(t_0) = 0$, taking the n-vector λ of the Lagrangian multipliers – adjoint variables – such that it satisfies the system of differential equations (Eq. 9):

$$\frac{d\lambda}{dt} = -H_x^T, \quad (9)$$

and the boundary conditions (Eq. 10)

$$\lambda(t_f) = -g_{x_f}^T, \quad (10)$$

Equation 8 reduces to Eq. 11:

$$\delta\bar{J} = -\int_{t_0}^{t_f} H_u \delta u dt. \quad (11)$$

Equations 7 and 11 define an isoperimetric problem of the calculus of variations, solution of which is given by Eqs. 12 and 13 (Gelfand and Fomin, 1963; Elsgolts, 1977):

$$\delta u = \frac{1}{2\nu} W^{-1} H_u^T, \quad (12)$$

$$\nu = \frac{1}{2K}\left\{\int_{t_0}^{t_f} H_u W^{-1} H_u^T dt\right\}^{1/2}, \quad (13)$$

where ν is the Lagrangian Multiplier corresponding to the constraint (Eq. 7).

Thus, from Eqs. 11 to 13, it follows Eq. 14:

$$\delta\bar{J} = \delta J = -K\left\{\int_{t_0}^{t_f} H_u W^{-1} H_u^T dt\right\}^{1/2}, \quad (14)$$

and the new value of the performance index will be smaller than the previous ones.

The step-by-step computing procedure to be used in the steepest descent method is summarized as:

- Integrate the differential equations (Eq. 1) from the initial point x_0 at t_0 to an unspecified final point $x(t_f)$ at t_f, with the nominal control $u^0(t),\ t_0 \le t \le t_f$;
- Integrate the adjoint differential equations (Eq. 9) from t_f to t_0, with the terminal conditions (Eq. 10);
- Compute the Lagrangian multiplier ν from Eq. 13;
- Compute the control 'correction' $\delta u(t),\ t_0 \le t \le t_f$ from Eq. 12;
- Obtain a new nominal control by using $u^1(t) = u^0(t) + \delta u(t)$, and repeat the process until the integral $\int_{t_0}^{t_f} H_u W^{-1} H_u^T dt$ tends to zero or other convergence criterion is satisfied.

It should be noted that as the nominal control $u^n(t)$ approaches the optimal one $u^*(t)$, the integral $\int_{t_0}^{t_f} H_u W^{-1} H_u^T dt$ approaches zero and the Lagrangian multiplier ν tends to zero; thus, the control 'correction' $\delta u(t)$, obtained from Eq. 12, can become too large and the process diverges. In order to avoid this drawback, the step size in control space K must be redefined. The following heuristic approach is proposed: suppose that K^m is the step size in control space at the m-th iterate, if $\int_{t_0}^{t_f} H_u W^{-1} H_u^T dt < L$, where L is a critical value for this integral, then K is redefined as $K^{m+1} = \beta K^m$, with $0 < \beta < 1$ as a reduction factor. L and β must be chosen by the user of the algorithm. Numerical experiments have shown that this approach provides good results.

SECOND VARIATION METHOD

The direct method based upon the second variation is also an iterative method used for computing a m-vector of control variables $u(t),\ t_0 \le t \le t_f$, which minimizes a scalar performance index in an optimization problem (Bullock and Franklin, 1967; Longmuir and Bohn, 1969; Bryson and Ho, 1975; Imae, 1998). In the present work, this method will be simply referred to as a second variation method.

The second variation method is developed for a Bolza problem of optimal control, with constrained final state and fixed terminal times. The generalized Riccati transformation as proposed by Longmuir and Bohn (1969) is applied for solving the linear two-point boundary value problem, associated

with the accessory minimization problem obtained from the second variation of the augmented performance index of the original optimization problem. Therefore, the algorithm here presented is different from the one described by Bullock and Franklin (1967), which was developed for a Mayer problem and involves a different set of transformation matrices for solving the linear two-point boundary value problem associated with the accessory minimization problem. The present algorithm is also different from the one proposed by Bryson and Ho (1975), since it does not include the stopping condition for time-free problems, which are treated by a transformation approach as mentioned. However, the algorithm is closely based on the approach for second variation methods described by Longmuir and Bohn (1969) and includes an important modification adopted by Bullock and Franklin (1967) in their algorithm in order to assure the convergence and to satisfy Legendre's condition.

For completeness, a brief description of the development of the second variation method is presented. Consider the system of differential equations (Eq. 15):

$$\frac{dx_i}{dt} = f_i(x,u), \quad i = 1,...,n\,, \tag{15}$$

where, x is a n-vector of state variables and u is a m-vector of control variables, $f(.): R^n \times R^m \rightarrow R^n$, $f_i(.)$ e $\partial f_i/\partial x_j$, i and j =1,...,n are defined and continuous on $R^n \times R^m$. It is assumed that there are no constraints on the state or control variables.

The problem consists in determining the control $u^*(t)$, which transfers the system (Eq. 15) from the initial conditions (Eq. 16):

$$x(t_0) = x_0, \tag{16}$$

to the final conditions at t_f (Eq. 17):

$$\psi(x(t_f)) = 0, \tag{17}$$

and minimizes the performance index (Eq. 18):

$$J[u] = g(x(t_f)) + \int_{t_0}^{t_f} F(x,u)\,dt\,, \tag{18}$$

with $g: R^n \rightarrow R$ and $\partial g/\partial x_i$, i =1,...,n, continuous on R^n . $F(.)$ and $\partial F/\partial x_i$, i =1,...,n are also defined and continuous on $R^n \times R^m$, and $\psi: R^n \rightarrow R^q$, $q<n$, ψ_i (.) and $\partial \psi_i/\partial x_j$, i=1,...,$q$, and j=1,...,n are defined and continuous on R^n. Furthermore, it is assumed that the matrix $\left[\frac{\partial \psi}{\partial x}\right]$ has maximum rank.

The second variation method is an extension of the steepest descent one already presented and it is also based on the classic calculus of variations. The main difference is the inclusion of the second-order terms in the expansion of the augmented performance index about a nominal solution. By virtue of this fact, the method is also known as second-order gradient method (Bryson and Ho, 1975).

Introducing the n-vector λ of Lagrangian multipliers – adjoint variables – and the q-vector μ of Lagrangian multipliers, the augmented performance index is formed by Eq. 19:

$$\overline{J} = g(x(t_f)) + \mu^T \psi(x(t_f)) + \int_{t_0}^{t_f} \left[-H + \lambda^T \dot{x}\right] dt\,. \tag{19}$$

Here, the Hamiltonian function H is defined as $H(x,\lambda,u) = -F(x,u) + \lambda^T f(x,u)$.

In order to derive the algorithm of the second variation method, one proceeds as in the steepest descent method. Let $u^0(t), t_0 \le t \le t_f$ be an arbitrary starting approximation of the control $u^*(t), t_0 \le t \le t_f$; $x^0(t), t_0 \le t \le t_f$, the corresponding trajectory, obtained by integrating the system (Eq. 1) and μ^0 be an arbitrary starting approximation of the Lagrange multiplier μ. And, $u^1(t) = u^0(t) + \delta u(t)$, $t_0 \le t \le t_f$, be the second iterate such that $J[u^1] \le J[u^0]$. The control variation $\delta u(t)$, $t_0 \le t \le t_f$ causes perturbations in the state vector $x(t)$, as well in the adjoint vector $\lambda(t)$ and in the Lagrange multiplier μ, so that the nominal values of $x(t)$, $\lambda(t)$ and μ, corresponding to the control $u^1(t)$, $t_0 \le t \le t_f$ can be expressed as Eq. 20:

$$\begin{aligned} &x^1(t) = x^0(t) + \delta x(t),\ t_0 \le t \le t_f \\ &\lambda^1(t) = \lambda^0(t) + \delta\lambda(t),\ t_0 \le t \le t_f \\ &\mu^1 = \mu^0 + \delta\mu. \end{aligned} \tag{20}$$

It is assumed that the perturbations δx, $\delta\lambda$, δu and $\delta\mu$ are small. Therefore, the change in $\overline{J}$, which is assumed to be twice differentiable, to second order in the perturbations, is given by Eq. 21 (Longmuir and Bohn, 1969):

$$\begin{aligned} \Delta\overline{J} = {} & \left(g_x + \mu^T \psi_x\right)\delta x(t_f) + \delta\mu^T \psi + \lambda^T \delta x \Big|_{t_0}^{t_f} \\ & + \int_{t_0}^{t_f} \left[\left(-\dot{\lambda}^T - H_x\right)\delta x + \left(\dot{x}^T - H_\lambda\right)\delta\lambda - H_u \delta u\right] dt \\ & + \frac{1}{2}\delta x(t_f)^T \left(g_{xx} + \mu^T \psi_{xx}\right)\delta x(t_f) + \delta\mu^T \psi_x\, \delta x(t_f) \\ & + \int_{t_0}^{t_f} \left(-\frac{1}{2}\delta x^T H_{xx}\delta x - \delta x^T H_{x\lambda}\delta\lambda - \delta x^T H_{xu}\delta u \right. \\ & \left. - \delta\lambda^T H_{\lambda u}\delta u - \frac{1}{2}\delta u^T H_{uu}\delta u + \delta\lambda^T \delta\dot{x}\right) dt. \end{aligned} \tag{21}$$

Considering that the state equations (Eq. 15) is as in Eq. 22:

$$\frac{dx}{dt} = H_{\lambda}^{T}, \tag{22}$$

with the initial conditions $x(t_0)=x_0$ is satisfied, as well as the adjoint equations (Eq. 23)

$$\frac{d\lambda}{dt} = -H_{x}^{T}, \tag{23}$$

with the boundary conditions (Eq. 24)

$$\lambda(t_f) = -(g_x + \mu^T \psi_x)^T; \tag{24}$$

and taking Eq. 25

$$\psi_x \delta x(t_f) = -\psi(x(t_f)), \tag{25}$$

Eq. 21 reduces to 26:

$$\begin{aligned}\Delta \bar{J} = {} & \frac{1}{2}\delta x(t_f)^T \left(g_{xx} + \mu^T \psi_{xx}\right)\delta x(t_f) \\ & + \int_{t_0}^{t_f} \left(-H_u \delta u - \frac{1}{2}\delta x^T H_{xx}\delta x - \delta x^T H_{x\lambda}\delta\lambda - \delta x^T H_{xu}\delta u \right. \\ & \left. - \delta\lambda^T H_{\lambda u}\delta u - \frac{1}{2}\delta u^T H_{uu}\delta u + \delta\lambda^T \delta\dot{x} \right) dt,\end{aligned} \tag{26}$$

since $\delta x(t_0) = 0$. Notice that $\delta x(t)$ satisfies the linear perturbation equation (Eq. 27):

$$\delta\dot{x} = H_{\lambda x}\delta x + H_{\lambda u}\delta u. \tag{27}$$

The change in the performance index, including the second order terms, can be written as Eq. 28:

$$\begin{aligned}\Delta \bar{J} = {} & \frac{1}{2}\delta x(t_f)^T \Phi_{xx}\delta x(t_f) \\ & + \int_{t_0}^{t_f} \left(-H_u \delta u - \frac{1}{2}\delta x^T H_{xx}\delta x \right. \\ & \left. - \delta x^T H_{xu}\delta u - \frac{1}{2}\delta u^T H_{uu}\delta u \right) dt,\end{aligned} \tag{28}$$

Where (Eq. 29):

$$\Phi_{xx} = g_{xx} + \mu^T \psi_{xx}. \tag{29}$$

The problem of determining $\delta u(t)$ is then defined by the following new problem of optimal control, known as accessory problem (Bullock and Franklin, 1967): determine $\delta u(t)$, $t_0 \le t \le t_f$ to minimize the performance index $\Delta \bar{J}$ defined by Eq. 28, subject to Eqs. 30 to 32:

$$\delta\dot{x} = H_{\lambda x}\delta x + H_{\lambda u}\delta u, \tag{30}$$

$$\delta x(t_0) = 0, \tag{31}$$

$$\psi_x \delta x(t_f) = -\psi(x(t_f)). \tag{32}$$

Following the Pontryagin's Maximum Principle (Pontryagin *et al.*, 1962), the *n*-vector *p* of adjoint variables is introduced and the Hamiltonian function $\tilde{H}(\delta x, p, \delta u)$ is formed by using Eq. 30:

$$\begin{aligned}\tilde{H}(\delta x, p, \delta u) = {} & \left(H_u \delta u + \frac{1}{2}\delta x^T H_{xx}\delta x + \delta x^T H_{xu}\delta u + \frac{1}{2}\delta u^T H_{uu}\delta u \right) \\ & + p^T \left(H_{\lambda x}\delta x + H_{\lambda u}\delta u \right).\end{aligned} \tag{33}$$

The optimal control $\delta u^*(t)$ (Eq. 34) must be chosen to maximize the Hamiltonian $\tilde{H}$ (Eq. 33):

$$\delta u^* = -H_{uu}^{-1}[H_u{}^T + H_{ux}\delta x + H_{u\lambda}\, p]. \tag{34}$$

The adjoint vector $\boldsymbol{p}(t)$ must satisfy the differential Eq. 35:

$$\dot{p} = -H_{x\lambda}p - (H_{xx}\delta x + H_{xu}\delta u), \tag{35}$$

with the boundary conditions (Eq. 36):

$$p(t_f) = -\Phi_{xx}\delta x(t_f) - \psi_x^T \sigma, \tag{36}$$

where σ is the Lagrangian multiplier corresponding to the constraint (Eq. 32). Note that Equations 35 and 36 are the linear perturbation ones corresponding to Eqs. 23 and 24, respectively; thus, $p(t)=\delta\lambda(t)$ and $\sigma = \delta\mu$.

Substituting $\delta\mu^*(t)$ into Eqs. 30 and 35, the following two-point boundary value problem is obtained (Eq. 37):

$$\begin{aligned}\delta\dot{x} &= A\delta x + B\delta\lambda + D \\ \delta\dot{\lambda} &= C\delta x - A^T\delta\lambda + E,\end{aligned} \tag{37}$$

where the matrices *A, B, C, D, E* are given by Eq. 38:

$$\begin{aligned}A &= H_{\lambda x} - H_{\lambda u}H_{uu}^{-1}H_{ux} \\ B &= -H_{\lambda x}H_{uu}^{-1}H_{u\lambda} \\ C &= H_{xu}H_{uu}^{-1}H_{ux} - H_{xx} \\ D &= -H_{\lambda u}H_{uu}^{-1}H_{u} \\ E &= H_{xu}H_{uu}^{-1}H_{u},\end{aligned} \tag{38}$$

with the boundary conditions (Eqs. 39 to 41):

$$\delta x(t_0) = 0, \tag{39}$$

$$\psi_x \delta x(t_f) = -k\psi(x(t_f)), \tag{40}$$

$$\delta\lambda(t_f) = -\Phi_{xx}\delta x(t_f) - \psi_x^T \delta\mu, \tag{41}$$

with $0 < k < 1$.

Equation 40 means that on each step, "partial corrections" are obtained. The parameter k must be chosen by the user of the algorithm. Numerical experiments show that this parameter can be used to avoid the method diverges.

The solution of the two-point boundary value problem above defined is obtained through a backward-sweep method, which uses the generalized Riccati transformation (Longmuir e Bohn, 1969; Bryson and Ho, 1975), as in Eqs. 42 and 43:

$$\delta\lambda(t) = R(t)\delta x(t) + L(t)\delta\mu + s(t), \tag{42}$$

$$0 = L^T(t)\delta x(t) + Q(t)\delta\mu + r(t), \tag{43}$$

where:

R is a $n \times n$ symmetric matrix,
L is a $n \times q$ matrix,
Q is a $q \times q$ symmetry matrix,
s is a $n \times 1$ matrix,
and r is a $q \times 1$ matrix.

In order that Eqs. 42 and 43 be consistent with Eqs. 37 to 41, the Riccati coefficients must satisfy the following differential equations (44 to 48):

$$-\dot{R} = RA + A^T R + RBR - C, \tag{44}$$

$$-\dot{L} = (A^T + RB)L, \tag{45}$$

$$-\dot{Q} = L^T BL, \tag{46}$$

$$-\dot{s} = (A^T + RB)s + RD - E, \tag{47}$$

$$-\dot{r} = L^T(D + Bs), \tag{48}$$

with the boundary conditions (49 to 53):

$$R(t_f) = -\Phi_{xx}, \tag{49}$$

$$L(t_f) = -\psi_x^T, \tag{50}$$

$$Q(t_f) = 0, \tag{51}$$

$$s(t_f) = 0, \tag{52}$$

$$r(t_f) = -k\,\psi. \tag{53}$$

The step-by-step computing procedure to be used in the second variation (or second order gradient) method is summarized as follows:

- Integrate the differential equations (15) from the initial point x_0 at t_0 to an unspecified final point $x(t_f)$ at t_f, with the starting nominal control $u^0(t), t_0 \le t \le t_f$;
- Choose a starting Lagrange multiplier μ^0;
- Integrate the adjoint differential equations (23) from t_f to t_0, with the boundary conditions (24);
- Compute the partial derivatives of the Hamiltonian function H - H_u, H_{uu}, H_{xu}, $H_{x\lambda}$, H_{xx} and $H_{\lambda u}$;
- Integrate Eqs. 44 to 48 backward from t_f to t_0, with the boundary conditions defined by Eqs. 49 to 53. Compute $R(t)$, $L(t)$, $Q(t)$, $s(t)$ and $r(t)$;
- Compute $\delta\mu = -Q(t_0)^{-1}[L^T(t_0)\delta x(t_0) + r(t_0)]$;
- Integrate the linear perturbation equation $\delta\dot{x} = (A+BR)\delta x + BL\delta\mu + Bs + D$, obtained from Eqs. 37 and 42;
- Compute $\delta\lambda(t)$ using Eq. 42;
- Compute δu^* using Eq. 34;
- Compute the new control $u^1(t) = u^0(t) + \delta u(t)$, $t_0 \le t \le t_f$, and the Lagrange multiplier $\mu^1 = \mu^0 + \delta\mu$;
- Test the convergence. Repeat the process until it converges.

It should be noted that the algorithm of the second variation method diverges if the Legendre condition $H_{uu} < 0$, computed for the nominal solution over the whole time interval $t_0 \le t \le t_f$, is not satisfied. However, by adding a term $\frac{1}{2}\|\delta u\|^2 = \frac{1}{2}\int_{t_0}^{t_f} \delta u^T W_2 \delta u dt$, which further constraints the control effort, the resulting functional $\Delta\bar{J}$ satisfies the Legendre condition if the m × m matrix W_2 is chosen large enough. Therefore, H_{uu} must be replaced by $H_{uu} + W_2$ in the algorithm already described. Such modification is equivalent to the constraint (Eq. 7) introduced in the steepest descent method (Bullock and Franklin, 1967).

THE COMBINED ALGORITHM

As discussed in the book by Bryson and Ho (1975), the main characteristic of the steepest descent method is the great improvements in the first few iterations, but it has poor convergence as the optimal solution is approached. On the other hand, the second variation method presents excellent convergence characteristics as the optimal solution is approached, but it requires the nominal solution to be 'convex'. Although the modification introduced in the algorithm provides the required convexity conditions, numerical experiments have shown that the accuracy of the solution is very sensitive to the choice of matrix W_2, if the first approximation of the control is poor.

Considering these remarks, an algorithm that combines the best characteristics of each method is implemented: at the first step, the steepest descent method is applied to improve the nominal solution to a point where it is convex; then, at the second one, the second variation method is applied, using the value of u obtained as the solution of the first algorithm, as the initial guess of the second one. The point where the algorithm commutes from the steepest descent to the second order method is defined by the user and is based on the value desired for the integral:

$$\int H_u W^{-1} H_u^T dt.$$

The resulting algorithm will be simply referred to as the combined algorithm and its implementation is a combination of the step-by-step computing procedure. As it will be later presented, the combined algorithm based on the simplified versions of the steepest descent and second variation methods described in the preceding sections provides good results, which have motivated its application to the analysis of optimal low-thrust limited-power transfers between coplanar circular orbits.

SIMPLE NUMERICAL EXAMPLES

In order to clarify some of the algorithms aspects of the methods described, two classical examples were considered: the Zermelo and the Brachistochrone problems.

Zermelo problem

It is a classical minimum-time navigation problem (Leitmann, 1981). Consider a boat moving with velocity $\boldsymbol{v}$ of constant magnitude $\boldsymbol{v}$ =1 relative to a stream of constant velocity $\boldsymbol{s}$. The problem consists in determining the steering program that transfers the boat from a given initial position to a given terminal position in minimum time. The statement of the optimization problem is as follows: determine the control $\theta(t)$ that transfers the system described by differential equations (54):

$$\frac{dx_1}{dt} = \cos\theta + s \qquad \frac{dx_2}{dt} = \sin\theta \qquad \frac{dx_3}{dt} = 1, \tag{54}$$

from the initial (55)

$$x_i(t_0) = x_i^0, i = 1,2,3, \tag{55}$$

to the final conditions (56)

$$x_1(t_f) = x_1^f \qquad x_2(t_f) = x_2^f, \tag{56}$$

and minimizes the performance index (57)

$$J = x_3(t_f). \tag{57}$$

In order to solve this time-free problem by means of the algorithms of the steepest descent and the second variation methods, a transformation approach must be used. This time-free problem will be transformed in a new time-fixed problem through a simple device (Hussu, 1972; Quintana and Davison, 1973).

Introducing new independent τ, $\tau \in [0,1]$ and auxiliary control variables α through the Eq. 58

$$t = \alpha\tau, \tag{58}$$

the time-free problem defined by Eqs. 54 to 57 is converted to a time-fixed problem with state equations given by (59):

$$\frac{dx_1}{d\tau} = \alpha(\cos\theta + s) \qquad \frac{dx_2}{d\tau} = \alpha\sin\theta \qquad \frac{dx_3}{d\tau} = \alpha \tag{59}$$

In order to apply the algorithm of the steepest descent method derived for a Mayer problem with free final state, the penalty function method (Hestenes, 1969; O'Doherty and Pierson, 1974) must be used. The new time-fixed problem is then defined by the state equations (59), the initial conditions (55), and a new performance index obtained from Eqs. 56 and 57:

$$J = x_3(1) + \left[k_1\left(x_1(1) - x_1^f\right)^2 + k_2\left(x_2(1) - x_2^f\right)^2\right], \tag{60}$$

where $k_1, k_2 >> 1$ must be chosen by the user of the algorithm. In the penalty function method as described by O'Doherty and Pierson (1974), the penalty constants k_1 and k_2 are progressively increased; however, for simplicity, we will take fixed

value for these constants as adopted by Lasdon *et al.* (1967) in the conjugate gradient method.

Both methods use the partial derivatives of the Hamiltonian function H, which is given by $H=\alpha[\lambda_1(\cos\theta+s)+\lambda_2\sin\theta+\lambda_3]$.

As an example, consider s=0.2, the initial (0,0) and final point (4,1). The exact solution (with seven significant figures) of this problem is t_f=3.456512. The first approximation of the control variables is chosen as Eqs. 61 and 62 for both algorithms:

$$\theta(\tau) = 0.5, \tau \in [0,1], \tag{61}$$

$$\alpha(\tau) = 2.0, \tau \in [0,1], \tag{62}$$

The numerical results are shown in Table 1 and Figs. 1 to 3. In both cases, the convergence criterion is given by $|IP^{n+1}\text{-}IP^{n}| < 1.25\times10^{-7}$.

Table 1. Numerical results for Zermelo problem.

Optimization method	J	$x_1(1)$	$x_2(1)$	Number of iterations	$\lvert\Delta IP\rvert$
Steepest descent	3.4556	3.9982	1.0023	69	8.41×10^{-4}
Second variation	3.4565	3.9999	0.9999	5	–
Combined algorithm	3.4565	3.9999	0.9999	6	–

For the steepest descent method, the numerical results are obtained with the following set of parameters: weights of the penalty function, k_1=k_2=500; reduction factor for the step size in control space, β=0.50; initial step size in control space K_0=0.05; critical value of the integral $\int_{t_0}^{t_f} H_u W^{-1} H_u^T dt$, L=1000. For the second variation method, the numerical results are presented with the following set of parameters: reduction factor for partial corrections of the terminal constraint defined in Eq. 40, k=1.0; reduction factor for δu and $\delta\mu$ corrections ($0< \varepsilon \leq 1$), ε=1.0; elements of the diagonal matrix W_2, $W_{ii,2}$= -2000. For simplicity, W_2 has been chosen as a diagonal matrix. For this set of parameters, the second variation method requires less iterations than the steepest descent method and the accuracy of the solution is much better. The last column provides a difference between the value of the performance index obtained through the algorithms and the exact solution. In Fig 1, the rates of convergence for steepest descent and second variation methods are presented. Figures 2 and 3 show the time history of the optimal control and trajectory for both methods.

The results of the combined algorithm are also presented in Table 1, taking the same parameters already defined. The algorithm provides a very accurate solution in few iterations. In this example, the performance is the same of the second variation method. We could note that the algorithm will be more effective in the next example, in which the Brachistochrone problem is solved.

The number of iterations obtained in all algorithms is closely related to the choice of the starting approximation, which is "poor". The number of iterations can be reduced if a good first approximation is chosen. For instance, when one takes the starting approximation as $\theta(\tau)$=0.3 and $\alpha(\tau)$=3.0, $\tau\in[0,1]$, the number of iterations obtained by the steepest descent method with the same set of parameters previously defined will be 32, which is much better than 69.

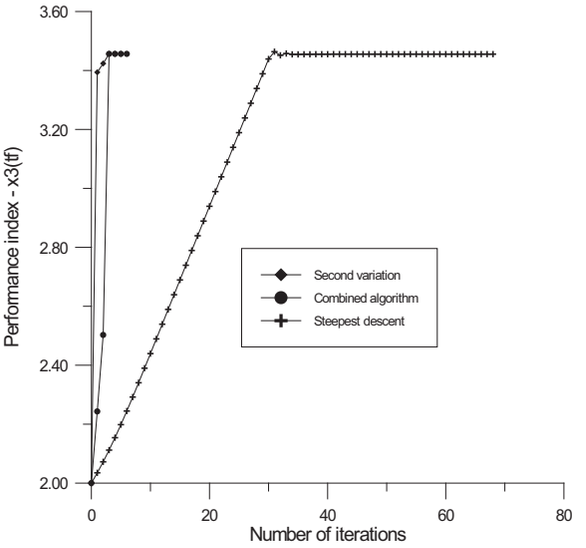


Figure 1. Convergence rates of the steepest descent and second variation methods for Zermelo problem.

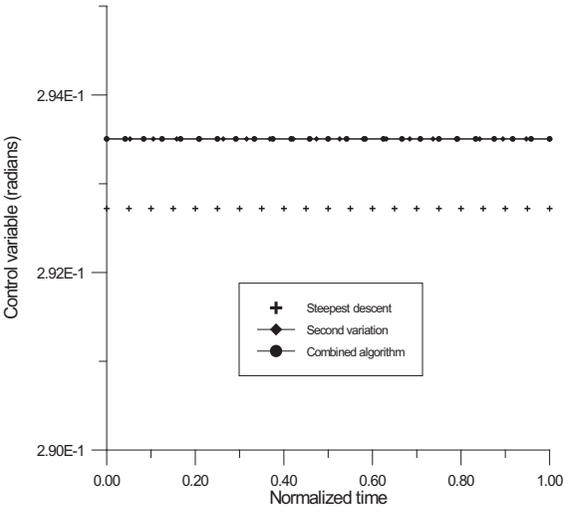


Figure 2. Time history of optimal control for Zermelo problem.

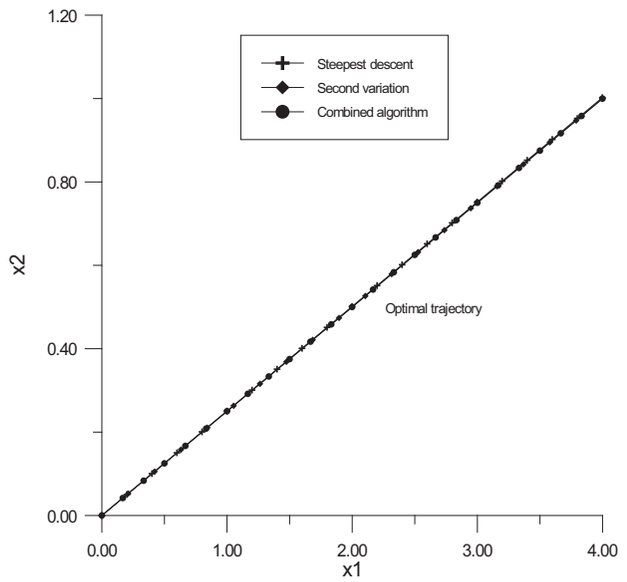


Figure 3. Optimal trajectory for Zermelo problem.

Brachistochrone problem

One of the most famous issue in the Mathematics history is the Brachistochrone problem, which consists of determining the curve $y(x)$ that a m mass particle will slide from point A to a lower B, without friction, in the minimum time. The statement of the problem is as follows (Williamson and Tapley, 1972; Bryson and Ho, 1975). Determine the control $u(t)$ that transfers the system described by the differential equations:

$$\frac{dx_1}{dt} = \sqrt{x_2}\cos u \qquad \frac{dx_2}{dt} = \sqrt{x_2}\sin u \qquad \frac{dx_3}{dt} = 1 \tag{63}$$

from the initial conditions:

$$x_i(t_0) = x_i^{\,0}, i = 1, 2, 3, \tag{64}$$

to the final ones:

$$x_1(t_f) = x_1^{\,f} \qquad x_2(t_f) = \text{free}, \tag{65}$$

and minimizes the performance index:

$$J = x_3(t_f). \tag{66}$$

In order to solve this time-free problem by means of the algorithms of the steepest descent and the second variation methods, one proceeds as described in the previous section. Introducing a new independent variable τ, $\tau \in [0,1]$ and an auxiliary control variable α through the equation 67

$$t = \alpha\tau, \tag{67}$$

the time-free problem defined by Eqs. 63 to 66 is converted to a time-fixed problem with state equations given by Eq. 68:

$$\frac{dx_1}{d\tau} = \alpha\sqrt{x_2}\cos u \qquad \frac{dx_2}{d\tau} = \alpha\sqrt{x_2}\sin u \qquad \frac{dx_3}{d\tau} = \alpha\,. \tag{68}$$

The penalty function method (Hestenes, 1969; O'Doherty and Pierson, 1974) must be used in order to apply the algorithm of the steepest descent method derived for a Mayer problem with free final state. The new time-fixed problem is then defined by the state equations (68), the initial conditions (64), and a new performance index obtained from Eqs. 65 and 66:

$$J = x_3(1) + k_1\left(x_1(1) - x_1^{\,f}\right)^2, \tag{69}$$

where $k_1 >> 1$ must be chosen by the user of the algorithm.

Both methods use the partial derivatives of the Hamiltonian function H, which is given by Eq. 70:

$$H = \alpha\left[\sqrt{x_2}\left[\lambda_1\cos u + \lambda_2\sin u\right] + \lambda_3\right]. \tag{70}$$

As an example, consider the following boundary conditions (Williamson and Tapley, 1972):

$$x_1(0) = 0.015134313 \qquad x_2(0) = 0.059647112$$

$$x_3(0) = 0.5 \qquad x_1(1) = 1.0\,.$$

The exact solution to these boundary conditions is $x_2(t_f)$=0.634974 and $x_3(t_f)$ = 2.006941. The first approximation of the control variables is chosen as Eqs. 71 and 72 for the two algorithms.

$$u(\tau) = 0.5,\ \tau \in [0,1], \tag{71}$$

$$\alpha(\tau) = 1.5,\ \tau \in [0,1], \tag{72}$$

The numerical results are shown in Table 2. In both cases, the convergence criterion is given by $\left|IP^{n+1} - IP^{n}\right| < 1.0\times10^{-6}$.

The numerical results are obtained taking: $k_1 = k_2 = 1000$; $\beta = 0.95$; $K_0 = 0.02$; $L = 400$, for the steepest descent method, and $k = 0.25$; $\varepsilon = 0.75$; $W_{ii,2} = -2000$, for the second variation method. For this set of parameters, the second variation

method requires less iterations than the steepest descent method, but the accuracy of the solution is worse. In Fig 4, the rates of convergence for steepest descent and second variation methods are presented. For the steepest descent method, Fig. 4 shows an oscillatory behaviour, which is produced by the change of the step size in control space. Figures 5 and 6 shows the time history of the optimal control and trajectory for both methods. For the sake of simplicity, x_2- axis is upwards. The results for the combined algorithm are also presented in Table 2. It should be noted that the algorithm converges in less iterations and presents a good accuracy of the solution. The oscillatory behaviour in the first iterations shown in Fig. 4 is produced by the change of the step size in control space in the first stage of the algorithm (steepest descent method).

Table 2. Numerical results for Brachistochrone problem.

Optimization method	J	$x_1(1)$	$x_2(1)$	Number of iterations	$\|\Delta IP\|$
Steepest descent	2.0063	0.9993	0.6382	477	5.97×10^{-4}
Second variation	2.0363	0.9999	0.6463	59	2.94×10^{-2}
Combined algorithm	2.0070	1.0000	0.6293	20	6.00×10^{-5}

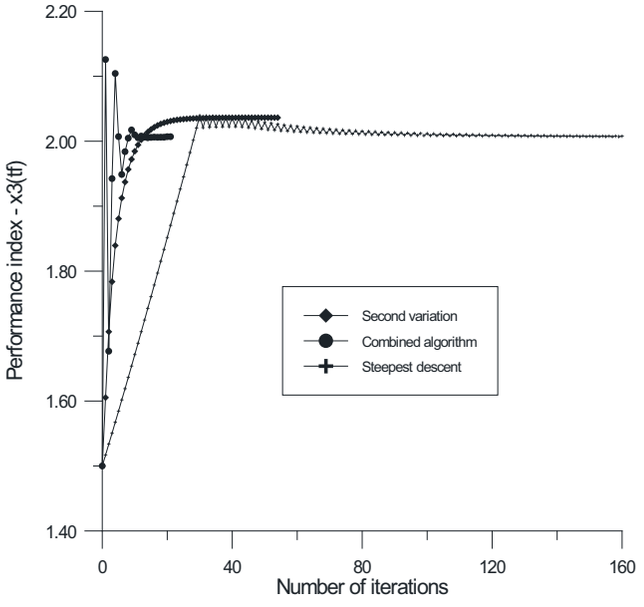


Figure 4. Convergence rate of the steepest descent method for Brachistochrone problem.

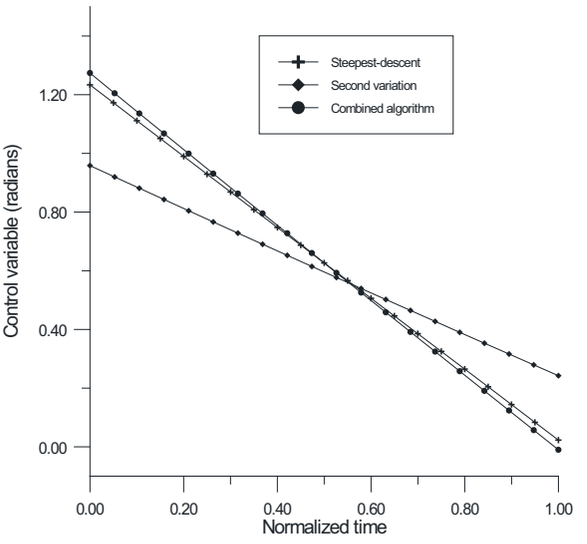


Figure 5. Time history of optimal control for Brachistochrone problem.

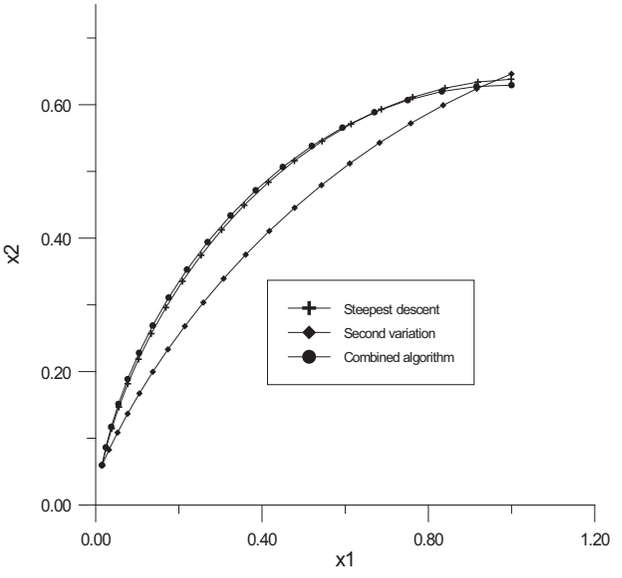


Figure 6. Optimal trajectory for Brachistochrone problem.

APPLICATION TO SPACE TRAJECTORIES OPTIMIZATION TRANSFER BETWEEN COPLANAR CIRCULAR ORBITS

Low-thrust limited power propulsion systems are characterized by low-thrust acceleration level and high specific impulse (Marec, 1976). The ratio between the maximum thrust and the gravity accelerations on the ground γ_{max}/g_0 is between 10^{-4} e 10^{-2}. For such system, the fuel consumption is described by the variable J defined as Eq. 73:

$$J = \frac{1}{2}\int_{t_0}^{t} \gamma^2 dt\,, \tag{73}$$

where γ is the magnitude of the thrust acceleration vector γ, used as a control variable. The consumption variable J is a monotonic decreasing function of the mass m of the space vehicle: $J = P_{\max}\left(\frac{1}{m} - \frac{1}{m_0}\right)$, where $P_{\max}$ is the maximum power and m_0 is the initial mass. The minimization of the final value J_f is equivalent to the maximization of m_f or the minimization of the fuel consumption.

The optimization problem concerning simple transfers (no rendezvous) between coplanar orbits is formulated as: at time t, the state of a space vehicle M is defined by the radial distance r from the center of attraction, the radial and circumferential components of the velocity, u and v, and the fuel consumption J. In the two-dimensional formulation, the state equations are given by Eq. 74:

$$\begin{aligned} \frac{du}{dt} &= \frac{v^2}{r} - \frac{\mu}{r^2} + R \\ \frac{dv}{dt} &= -\frac{uv}{r} + S \\ \frac{dr}{dt} &= u \\ \frac{dJ}{dt} &= \frac{1}{2}\left(R^2 + S^2\right), \end{aligned} \tag{74}$$

where, μ is the gravitational parameter, R e S are, respectively, radial and circumferential thrust acceleration vectors, from initial conditions t_0

$$u(0) = 0 \qquad v(0) = 1 \qquad r(0) = 1 \qquad J(0) = 0, \tag{75}$$

to final states in t_f

$$u(t_f) = 0 \qquad v(t_f) = \sqrt{\frac{\mu}{r_f}} \qquad r(t_f) = r_f\,, \tag{76}$$

such that J_f be minimum. The performance index is given by Eq. 77:

$$IP = J(t_f). \tag{77}$$

All variables are taken nondimensionally and we suppose there are no restrictions in the acceleration vector.

As an example of the application of the proposed method for orbit transfer problem described, consider Figs. 7 and 8, which present the consumption variable J as function of $\rho = \frac{r_f}{r_0} > 1$ and $\rho<1$, respectively, for nondimensional times of flight two to five.

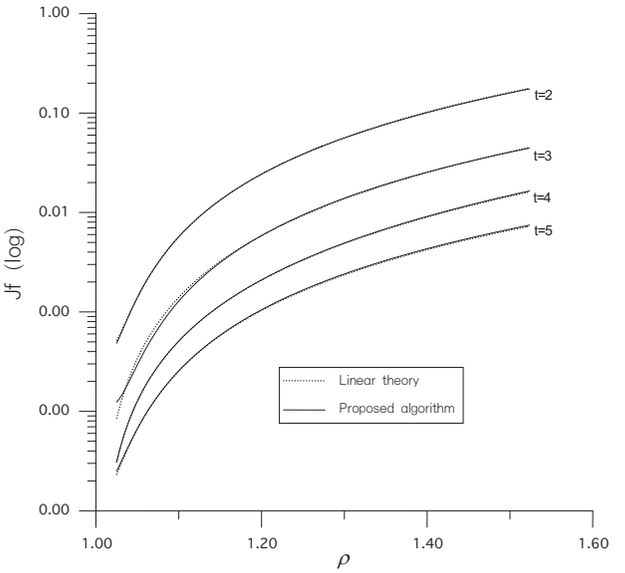


Figure 7. Consumption for ρ >1 and nondimensional time of flight from 2 to 5.

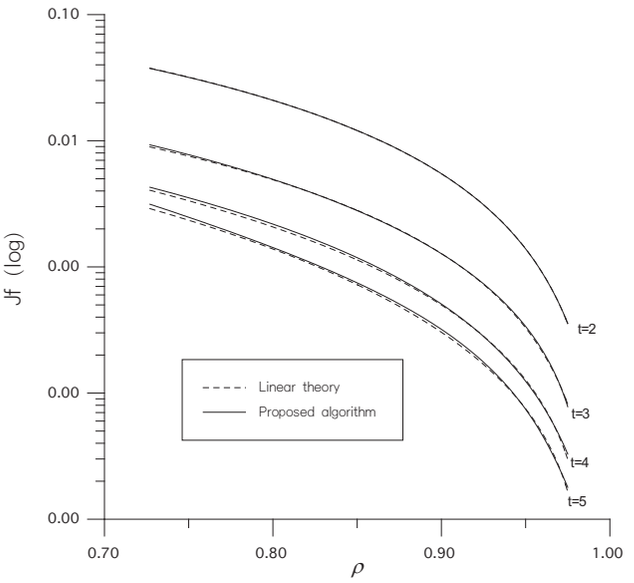


Figure 8. Consumption ρ >1 and non-dimensional time of flight from 2 to 5.

These figures present also a comparison between the results obtained with the proposal algorithm (combined algorithm) and a linear theory (Silva Fernandes and Golfetto, 2007). One can verify that the curves match almost precisely, presenting good agreement between the numerical and analytical results. The method provides a good approximation for the solution of low-thrust limited power transfer to coplanar circular orbits in a central Newtonian gravitational field problems. It can be noted that the bigger the difference between initial and final radii, the bigger is the fuel consumption.

In Figs. 9 and 10, respectively, one can verify the radial and circumferential acceleration evolutions for t_f - t_0 = 3 and ρ = 1.523 (Earth-Mars).

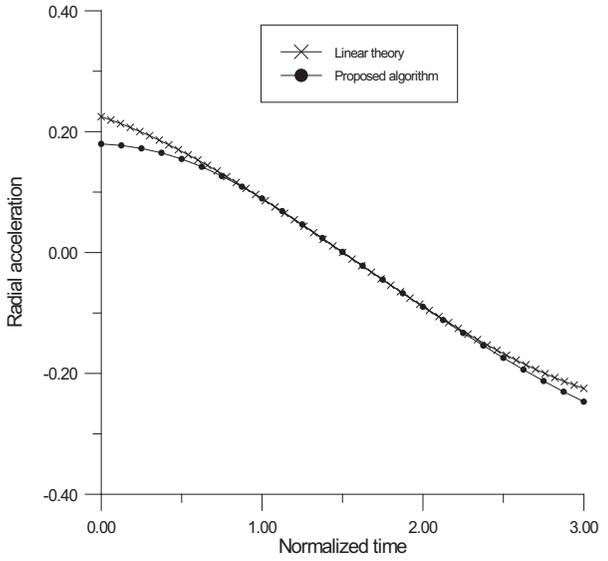


Figure 9. Radial acceleration history for ρ=1.523 and t_f-t_0= 3.

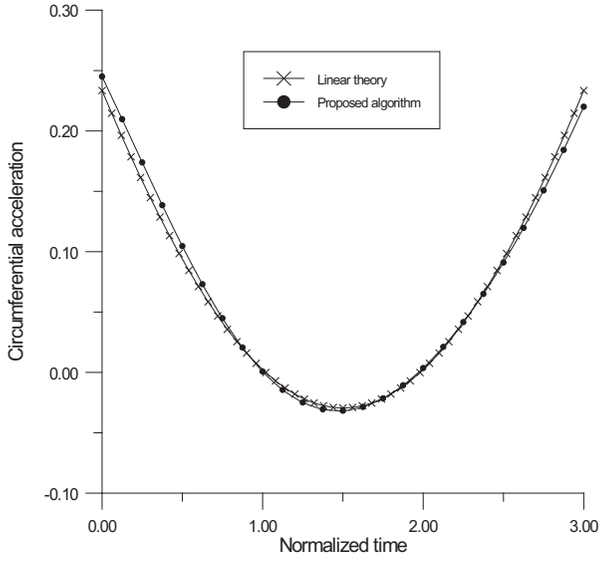


Figure 10. Circumferential acceleration history for ρ=1.523 and t_f-t_0= 3.

CONCLUSIONS

In this paper, two classic direct methods – steepest descent and second variation – for computing optimal trajectories are reviewed, and the main advantages and disadvantages of these methods are discussed. An algorithm that combines the good characteristics of each method is also presented. The algorithms are applied for solving two classic optimization problems: Zermelo and Brachistochrone problems. Finally, the proposed algorithm is used to solve the problem of optimization of space trajectories transference between coplanar circular orbits with a fixed time. It has been demonstrated the good approximation that the algorithm can provide.

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